Math 2040C Week 6

Invertibility and Isomorphic Vector spaces

Defn 3.53 A linear map $T \in L(V,W)$ is called **invertible** if $\exists S \in L(W,V)$ such that

$ST = I_V \in L(V,V)$, $TS = I_W \in L(W,W)$

are identity maps.

Such a S is called **an inverse of T**

Rmk Both $ST = I_V$ and $TS = I_W$ are needed. $ST = I_V \not\Rightarrow TS = I_W$

eg. Let $S, T : P(\mathbb{R}) \to P(\mathbb{R})$

$$Sp = p', \quad (Tp)(x) = \int_0^x p(t)\,dt$$

Then $ST = I_{P(\mathbb{R})}$ but $TS \neq I_{P(\mathbb{R})}$

Prop 3.54

An invertible linear map has a unique inverse

Pf Suppose $T \in L(V,W)$ is invertible and $S_1, S_2 \in L(W,V)$ are inverses. Then

$$S_1 = S_1 I_W = S_1(TS_2) = (S_1T)S_2 = I_V S_2 = S_2$$

Notation 3.55 For an invertible $T \in L(V,W)$, its unique inverse is denoted by T^{-1}.

Rmk If T is invertible, it is clear that T^{-1} is also invertible with $(T^{-1})^{-1} = T$.

The next proposition characterizes invertible linear maps

Prop 3.56 Let $T \in \mathcal{L}(V,W)$.

T is invertible $\Leftrightarrow$ T is bijective

(i.e. injective and surjective)

We simplify its proof in the book by using the following fact in set theory:

Fact Let $f: A \to B$ be a function between sets A and B. Then

$$f \text{ is bijective} \Leftrightarrow \exists g: B \to A \text{ such that } f \circ g = I_B,\ g \circ f = I_A$$

Pf of Prop 3.56

($\Rightarrow$) Simple exercise on set theory

($\Leftarrow$) Suppose T is bijective. Then by set theory, $\exists$ a function $S: W \to V$ s.t. $T \circ S = I_W$ and $S \circ T = I_V$.

We only need to show that S is linear:

Suppose $w_1, w_2 \in W$. Then

$$T(S(w_1) + S(w_2)) = T(S(w_1)) + T(S(w_2))$$
$$= w_1 + w_2$$

Apply S on both sides:

$$\Rightarrow S(w_1 + w_2) = S\left(T(S(w_1) + S(w_2))\right)$$
$$= (S \circ T)(S(w_1) + S(w_2))$$
$$= S(w_1) + S(w_2)$$

Similarly, $S(\lambda w) = \lambda S(w)$ $\forall w \in V, \lambda \in \mathbb{F}$

Hence, S is linear and so T is invertible

Isomorphic Vector Spaces

Defn 3.58 A invertible linear map is called an isomorphism (Greek: equal shape)
If there is an isomorphism $T: V \to W$
V and W are called isomorphic and is denoted by $V \cong W$

eg. Let $T_1, T_2, T_3 : P_2(\mathbb{R}) \to \mathbb{R}^3$ be defined by

$$T_1(a_0 + a_1 x + a_2 x^2) = (a_0, a_1, a_2)$$

$$T_2(p) = (p(0), p'(0), p''(0))$$

$$T_3(p) = (p(1), p(2), p(3))$$

are all isomorphisms.

Hence, $P_2(\mathbb{R})$ and $\mathbb{R}^2$ are isomorphic.

Thm 3.59 Two finite dim vector spaces over $\mathbb{F}$ are isomorphic if and only if they have equal dimension

Pf ($\Rightarrow$) Let $\dim V, \dim W < \infty$ and $T \in \mathcal{L}(V, W)$ is an isomorphism.

$$\begin{aligned} \text{Then } \dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &= \dim \{0_V\} + \dim W \\ &= \dim W \end{aligned}$$

($\Leftarrow$) If $\dim V = \dim W < \infty$.

Let $\{v_1, \cdots, v_n\}, \{w_1, \cdots, w_n\}$ be basis of V, W

Define $T \in \mathcal{L}(V, W)$ by $T(v_i) = w_i$

Then clearly T is an isomorphism (Ex)

Prop 3.60

Let $\alpha = \{v_1, \cdots, v_n\}$ be ordered basis of V

$\beta = \{w_1, \cdots, w_m\}$ be ordered basis of W

Then $M : L(V,W) \longrightarrow M_{m\times n}(\mathbb{F})$ defined by

$M(T) = M(T, \alpha, \beta)$ is an isomorphism

Pf ① **M is linear**: follows from formulas

$M(T_1 + T_2) = M(T_1) + M(T_2)$, $M(\lambda T) = \lambda M(T)$

② **M is injective**: Suppose $T \in \operatorname{null} M$, then

$M(T) = \mathbf{0}$ = zero matrix

$\Rightarrow$ its columns $M(T(v_i)) = \vec{0} \in \mathbb{F}^m$

$\therefore T(v_i) = 0w_1 + \cdots + 0w_m = 0_W \quad \forall i$

$\operatorname{span} \alpha = V \Rightarrow T(v) = 0 \quad \forall v \in V$

$\Rightarrow T = T_0$ and $\operatorname{null} M = \{T_0\}$

$\therefore$ M is injective

③ **M is surjective**: Suppose $A = (A_{ij}) \in M_{m\times n}(\mathbb{F})$

α is basis $\xRightarrow[\text{Prop 3.5}]{}$ $\exists$ unique $T \in L(V,W)$ s.t.

$$T(v_k) = \sum_{i=1}^{m} A_{ik} w_i \quad \forall k$$

Then $M(T) = A \Rightarrow M$ is surjective

Hence, M is an isomorphism

Rmk

① $\underbrace{L(V,W)}_{\text{Abstract}} \underset{M^{-1}}{\overset{M}{\rightleftarrows}} \underbrace{M_{m\times n}(\mathbb{F})}_{\text{Easy to understand, good for composition}}$ are isomorphic

$M_{m\times n}(\mathbb{F})$ has basis $\{E_{ij} : 1 \le i \le m, 1 \le j \le n\}$, where

$$(E_{ij})_{k\ell} = \begin{cases} 1 & \text{if } (k,\ell) = (i,j) \\ 0 & \text{if } (k,\ell) \ne (i,j) \end{cases}$$

(eg In $M_{2\times 3}(\mathbb{R})$ $E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$)

Let $T_{ij} = M^{-1}(E_{ij}) \in L(V,W)$. Then

$$T_{ij}(v_k) = \begin{cases} w_i & \text{if } k = j \\ \vec{0}_W & \text{if } k \ne j \end{cases}$$

$L(V,W)$ has basis $\{T_{ij} : 1 \le i \le m, 1 \le j \le n\}$

② $\dim L(V,W) = (\dim V)(\dim W)$

③ Similarly, $M: V \longrightarrow \mathbb{F}^n$ defined by

$$M(v) = M(v,\alpha) \text{ is an isomorphism}$$

Prop Let V, W be finite dim with ordered basis α, β respectively. Suppose $T \in L(V,W)$ is an isomorphism. Then $M(T,\alpha,\beta)$ is an invertible matrix

$$M(T^{-1},\beta,\alpha) = M(T,\alpha,\beta)^{-1} \quad ⊛$$

Matrix of inverse map   inverse of matrix

Pf Let $n = \dim V$. Since $V \cong W$,

$\dim W = \dim V = n \Rightarrow M(T,\alpha,\beta)$ is $n \times n$

Also $M(T^{-1},\beta,\alpha)\, M(T,\alpha,\beta)$

$= M(T^{-1} \circ T, \alpha, \alpha)$

$= M(I_V, \alpha, \alpha) = I_n \quad \Rightarrow ⊛$

Change of Basis

Suppose $\alpha = \{v_1, \cdots, v_n\}$, $\beta = \{w_1, \cdots, w_n\}$ be ordered bases of V

Let $M(I_V, \alpha, \beta) = Q = \begin{bmatrix} Q_{11} & \cdots & Q_{1n} \\ \vdots & \ddots & \vdots \\ Q_{n1} & \cdots & Q_{nn} \end{bmatrix} = ?$

Recall:

$$M(T,\alpha,\beta) = \begin{bmatrix} | & & | \\ M(T(v_1),\beta) & \cdots & M(T(v_n),\beta) \\ | & & | \end{bmatrix}$$

Put $T = I_V \Rightarrow Q = \begin{bmatrix} | & & | \\ M(v_1,\beta) & \cdots & M(v_n,\beta) \\ | & & | \end{bmatrix}$

j-th column $\Rightarrow v_j = \sum_{i=1}^{n} Q_{ij} w_j$

on j-th column

$Q = M(I_V, \alpha, \beta)$ is called the change of coordinates matrix from α to β.

Formulas

① If α, β are ordered basis of V, $I = I_V$

$$M(I,\alpha,\beta) = M(I,\beta,\alpha)^{-1} \qquad M(v,\beta) = M(I,\alpha,\beta)\, M(v,\alpha)$$

② If α, α' are ordered bases of V, β, β' are ordered bases of W, $T \in \mathcal{L}(V,W)$

$$\underbrace{M(T,\alpha',\beta')}_{\substack{\text{input } \alpha \\ \text{output } \beta}} = \underbrace{M(I_W,\beta,\beta')}_{\beta \to \beta'}\, \underbrace{M(T,\alpha,\beta)}_{\substack{\text{input } \alpha \\ \text{output } \beta}}\, \underbrace{M(I_V,\alpha',\alpha)}_{\alpha' \to \alpha}$$

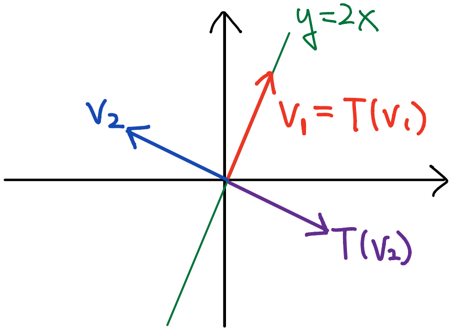
Note if $v \in V$, then

$$\underbrace{M(I_W,\beta,\beta')\, \underbrace{M(T,\alpha,\beta)\, \underbrace{M(I_V,\alpha',\alpha)\, M(v,\alpha')}_{M(v,\alpha)}}_{M(T(v),\beta)}}_{M(T(v),\beta') \;=\; M(T,\alpha',\beta')\, M(v,\alpha')}$$

eg Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection across the line $L: y = 2x$. Find $M(T, \beta, \beta)$, where $\beta = \{e_1, e_2\}$

Rmk T is linear!

Sol let $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in L$, $v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \perp L$



Then $T(v_1) = v_1$, $T(v_2) = -v_2$

Let $\alpha = \{v_1, v_2\}$

Then $M(T, \alpha, \alpha) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Since $v_1 = 1 e_1 + 2 e_2$

$v_2 = -2 e_1 + 1 e_2$

$$Q = M(T, \alpha, \beta) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$\therefore \quad M(T, \beta, \beta)$$
$$= Q\, M(T, \alpha, \alpha)\, Q^{-1}$$
$$= \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1}$$
$$= \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$$

eg. $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{7}{5} \end{bmatrix}$

Rmk Note $M(T, \alpha, \alpha)$ is a diagonal matrix.

T is called a diagonalizable operator.

Next topic